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Generalized vector quasi-equilibrium problems with set-valued mappings

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Abstract

This paper is devoted to study a new class of generalized vector quasi-equilibrium problems with set-valued mappings. By means of the Fan–KKM Theorem and lower semicontinuity with respect to cone order of the set-valued mapping, we obtain an existence result for this class of generalized vector quasi-equilibrium problems with set-valued mappings. Our result extents and improves some previous results.

Keywords: Generalized vector quasi-equilibrium problem; Solution existence; Fan–KKM Theorem; Lower semicontinuity with respect to cone order; Generalized vector 0-diagonally convex

1. Introduction

Equilibrium theory, including optimization problems, fixed pointed problems, variational inequalities, problems of the Nash equilibria and complementarity problems as special cases (see [1]), provides us a unified, natural, innovative and general framework for studying a wide class of problems arising in finance, economics, network analysis, transportation and elasticity. Recently, equilibrium problems involving set-valued mappings in ordered topological vector spaces are considered by many authors, for instance, Ansari et al. [2] and Ansari and Yao [3] studied generalized vector equilibrium problem with set-valued mapping, later Ansari and Flores-Bazan [4] considered the strong formulation of generalized vector equilibrium problem with set-valued mapping.

Let X, Y and Z be real topological vector spaces, $K \subset X$ and $D \subset Y$ nonempty subsets and $C \subset Z$ a closed convex cone with $\operatorname{int} C \neq \emptyset$, where $\operatorname{int} C$ denotes the interior of C. Let 2^D denote the family of all nonempty subsets of D and $T : K \to 2^D$ and $f : K \times D \times K \to Z$ be given. By means of Shioji's generalized Fan–KKM Theorem and Oettle's scalarization procedure, Chiang [5] studied the existence of solutions for generalized vector equilibrium problem:

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Find $(\widehat{x}, \widehat{y}) \in K \times T(\widehat{x})$ such that

$$f(\widehat{x}, \widehat{y}, u) \in (-\text{int } C)^c, \quad \forall u \in K,$$

where $(-\text{int } C)^c$ denotes the complement of $-\text{int } C$ in Z.

Let $G, H : D \times D \to 2^Z$ be two set-valued mappings and $C \subset Z$ a closed convex pointed cone with $\text{int } C \neq \emptyset$. By using the Fan–KKM Theorem and lower semicontinuity with respect to C of G, H, Fu [6] considered the existence of solutions for vector equilibrium problems:

(VEP1) Find $x \in D$ such that $G(x, y) + H(x, y) \subset Z \setminus (-\text{int } C), \forall y \in D$.
(VEP2) Find $x \in D$ such that $G(x, y) + H(x, y) \subset Z \setminus (-C \setminus \{0\}),\ \forall y \in D$.

In this paper, with the methods proposed by Chiang and Fu, we introduce and study a class of generalized vector quasi-equilibrium problem with set-valued mapping of finding $\widehat{x} \in K$ such that for each fixed $u \in K$, there exists $\widehat{y_u} \in T(\widehat{x})$ satisfying

$$F(\widehat{x}, \widehat{y_u}, u) \subset Z \setminus (-\text{int } C), \tag{1.1}$$

where $F : K \times D \times K \to 2^Z$ is a given set-valued mapping. The result obtained here for the existence of solutions of problem (1.1) extends main results of [5,6].

2. Preliminaries

In this section, we recall some basic concepts and results which will be used in the sequel. Let X and Y be topological spaces and $T : X \to 2^Y$ be a set-valued mapping.

T is said to be upper semicontinuous (in short, u.s.c.) at $x \in X$ if for each open set V of Y with $T(x) \subseteq V$, there is an open neighborhood U of x such that $T(z) \subset V$ for all $z \in U$. T is called u.s.c. on X if it is u.s.c. at each point in X. See [7].

T is said to be lower semicontinuous (in short, l.s.c.) at $x \in X$ if for each open set V of Y with $T(x) \cap V \neq \emptyset$, there is an open neighborhood U of x such that $T(z) \cap V \neq \emptyset$ for all $z \in U$. T is said to be l.s.c. on X if it is l.s.c. at each point in X. See [7].

T is said to closed if its graph $G(T) = \{(x, y) \in X \times Y : x \in K, y \in T(x)\}$ is a closed set in $X \times Y$. See [8].

Theorem 2.1 (*[8]*). *Let X and Y be topological spaces. If a set-valued mapping $T : X \to 2^Y$ is upper semicontinuous with compact values, then for every compact set $K \subset X$, the set $T(K) = \cup_{x \in K} T(x)$ is compact.*

Let D be a nonempty convex subset of a vector space X. A set-valued mapping $\varphi : D \to 2^X$ is called KKM-mapping if for each finite subset $\{x_1, \ldots, x_n\} \subset D$, one has

$$\text{co}\{x_1, \ldots, x_n\} \subset \cup_{i=1}^n \varphi(x_i),$$

where co$E$ *denotes the convex hull of a set E.*

Theorem 2.2. *(Fan–KKM Theorem). Let D be a nonempty convex subset of a Hausdorff topological vector space X and let $\varphi : D \to 2^X$ be a KKM-mapping. If*

(i) *$\varphi(x)$ is closed for each $x \in D$;*
(ii) *there exists $x_0 \in D$ such that $\varphi(x_0)$ is compact,*

then $\cap_{x \in D} \varphi(x) \neq \emptyset$.

Definition 2.1 (*[9]*). Let K and E be nonempty convex subsets of a vector space X with $E \subset K$, the set

$$\text{core}_K E = \{a \in E : E \cap (a, y] \neq \emptyset, \forall y \in K \setminus E\}$$

is called the core of E relative to K, where $(a, y] = \{x \in X : x = (1 - t)a + ty, t \in (0, 1]\}$.

Definition 2.2 (*[10]*). Let X and Y be real locally convex spaces, $C \subset Y$ a closed convex pointed cone and $D \subset X$ a nonempty subset. A set-valued mapping $T : D \to 2^Y$ is said to be lower semicontinuous with respect to C at $x \in D$

(in short, C-l.s.c.) if for each $y \in T(x)$ and for any open neighborhood V of y, there is an open neighborhood $U(x)$ of x such that

$$T(z) \cap (V + C) \neq \emptyset, \quad \forall z \in U(x) \cap D.$$

T is said to be C-l.s.c. on D if it is C-l.s.c. at each point in D.

It is evident that if T is l.s.c. at $x \in D$, then it is C-l.s.c. at x.

Definition 2.3. Let X, Y and Z be real topological vector spaces, $K \subset X$ and $D \subset Y$ nonempty convex subsets, and $C \subset Z$ a closed convex cone with int $C \neq \emptyset$.

(i) [6] A set-valued mapping $\varphi : K \to 2^Z$ is said to be C-convex if for any $x, y \in K$ and any $t \in [0, 1]$, one has

$$t\varphi(x) + (1-t)\varphi(y) \subset \varphi(tx + (1-t)y) + C;$$

(ii) Let $T : X \to 2^D$. A set-valued mapping $F : K \times D \times K \to 2^Z$ is called generalized vector 0-diagonally convex with respect to T if for any finite set $\{x_1, \ldots, x_n\} \subset K$ and any $x = \sum_{i=1}^n t_i x_i$ with $t_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^n t_i = 1$, there exists $y \in T(x)$ such that $\sum_{i=1}^n t_i F(x, y, x_i) \subset Z \setminus (-\text{int } C)$.

Note that generalized vector 0-diagonal convexity in Definition 2.3 is a natural extension of vector 0-diagonal convexity in [5].

3. Existence results

In this section, by virtue of the methods proposed in [5,6], we study the existence of solutions of problem (1.1).

Theorem 3.1. *Let X, Y and Z be real locally convex Hausdorff vector spaces, $K \subset X$ and $D \subset Y$ nonempty convex subsets, and $C \subset Z$ a closed convex pointed cone with* int $C \neq \emptyset$. *Let $T : X \to 2^D$ and $F : K \times D \times K \to 2^Z$ be two set-valued mappings satisfying the following assumptions:*

(i) *for every $x \in K$, $0 \in F(x, y, x)$ for all $y \in T(x)$ and there exists $y \in T(x)$ such that $F(x, y, x) \cap (-\text{int } C) = \emptyset$;*
(ii) *T is upper semicontinuous closed mapping with compact values;*
(iii) *F is generalized vector 0-diagonally convex with respect to T;*
(iv) *for each $u \in K$, the mapping $(x, y) : y \in T(x) \longmapsto F(x, y, u)$ is $(-C)$-l.s.c.;*
(v) *for each $(x, y) \in K \times T(x)$, the mapping $u \longmapsto F(x, y, u)$ is C-convex;*
(vi) *there exists a nonempty convex compact subset E of K such that for each $x \in E \setminus \text{core}_K E$, there exists an $a \in \text{core}_K E$ satisfying $F(x, y, a) \not\subseteq Z \setminus (-C)$ for all $y \in T(x)$.*

Then problem (1.1) has at least one solution.

To prove this theorem, we need the following lemma.

Lemma 3.1. *Let the hypotheses* (ii)–(iv)*in Theorem* 3.1 *hold and let E be any nonempty convex compact subset of K. If for every $x \in K$, there exists $y \in T(x)$ such that $F(x, y, x) \cap (-\text{int } C) = \emptyset$, then there is an $\widehat{x} \in E$ such that for every $u \in E$, there exists $\widehat{y}_u \in T(\widehat{x})$ satisfying (1.1).*

Proof. Define a set-valued mapping $S : E \to 2^E$ by

$$S(u) = \{x \in E : \exists y \in T(x) \ s.t. \ F(x, y, u) \subset Z \setminus (-\text{int } C)\}, \quad \forall u \in E.$$

Since for every $x \in K$, there exists $y \in T(x)$ such that $F(x, y, x) \cap (-\text{int } C) = \emptyset$, $S(u) \neq \emptyset$ for each $u \in E$. We can claim that S is a KKM mapping.

In fact, for any finite set $A = \{x_1, \ldots, x_n\} \subset E$ and any $x = \sum_{i=1}^n t_i x_i$ with $t_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^n t_i = 1$, by assumption (iii) in Theorem 3.1, there exists $y \in T(x)$ such that

$$\sum_{i=1}^n t_i F(x, y, x_i) \subset Z \setminus (-\text{int } C).$$

Consequently, there exists at least an $i_0 \in \{1, \ldots, n\}$ such that $F(x, y, x_{i_0}) \subset Z \setminus (-\text{int } C)$, which indicates that $x \in S(x_{i_0}) \subset \cup_{i=1}^n S(x_i)$. So, $\text{co}(A) \subset \cup_{i=1}^n S(x_i)$.

Next, we prove that $S(u)$ is closed for each $u \in E$.

Let $\{x_\alpha\}$ be any net in $S(u)$ converging to $x \in E$. Then for each α, there exists $y_\alpha \in T(x_\alpha) \subset T(E)$ such that

$$F(x_\alpha, y_\alpha, u) \subset Z \setminus (-\text{int}\, C). \tag{3.1}$$

By Theorem 2.1, we know that $T(E)$ is compact and then $\{y_\alpha\}$ has a convergent subnet. Without loss of generality, we can assume that $y_\alpha \to y \in T(E)$. By the closedness of T, one has

$$y \in T(x). \tag{3.2}$$

From (3.2), we can deduce that $x \in S(u)$. Indeed, for contradiction, assume that $F(x, y, u) \not\subseteq Z \setminus (-\text{int}\, C)$, then there exists $w \in F(x, y, u)$ such that $w \in -\text{int}\, C$. By assumption (iv) in Theorem 3.1, there exist open neighborhoods $U(x)$ and $V(y)$ of x and y, respectively, such that

$$F(z, v, u) \cap (-\text{int}\, C - C) = F(z, v, u) \cap (-\text{int}\, C) \neq \emptyset,$$

for all $z \in U(x) \cap K$ and $v \in T(z) \cap V(y)$. Since $x_\alpha \to x$ and $y_\alpha \to y$, there exists β such that

$$x_\alpha \in U(x) \cap E, \quad y_\alpha \in T(x_\alpha) \cap V(y), \quad \forall \alpha \geq \beta$$

and then $F(x_\alpha, y_\alpha, u) \cap (-\text{int}\, C) \neq \emptyset$, which contradicts (3.1).

By the Fan–KKM Theorem, we have $\cap_{u \in E} S(u) \neq \emptyset$, which indicates that the assertion of the lemma is true. □

Proof of Theorem 3.1. According to Lemma 3.1, there exists $\widehat{x} \in E$ such that for each $u \in E$, there exists $\widehat{y_u} \in T(\widehat{x})$ satisfying

$$F(\widehat{x}, \widehat{y_u}, u) \subset Z \setminus (-\text{int}\, C). \tag{3.3}$$

In order to prove the assertion of the theorem, it suffices to show that for each $u \in K \setminus E$, there also exists $\widehat{y_u} \in T(\widehat{x})$ satisfying (3.3). We assume for contradiction that there exists $\overline{u} \in K \setminus E$ such that

$$F(\widehat{x}, y, \overline{u}) \not\subseteq Z \setminus (-\text{int}\, C), \quad \forall y \in T(\widehat{x}).$$

Then for each $y \in T(\widehat{x})$ there exists $w_y \in F(\widehat{x}, y, \overline{u})$ such that $w_y \in -\text{int}\, C$.

If $\widehat{x} \in \text{core}_K E$, then $(\widehat{x}, \overline{u}] \cap E \neq \emptyset$. By assumption (i) of the theorem, we know that

$$0 \in F(\widehat{x}, y, \widehat{x}), \quad \forall y \in T(\widehat{x}).$$

Take arbitrarily $y \in T(\widehat{x})$ and $z \in (\widehat{x}, \overline{u}]$: $z = t\widehat{x} + (1-t)\overline{u}$ and $t \in [0, 1)$. It follows from the C-convexity of $F(\widehat{x}, y, \cdot)$ that

$$t0 + (1-t)w_y \in tF(\widehat{x}, y, \widehat{x}) + (1-t)F(\widehat{x}, y, \overline{u}) \subset F(\widehat{x}, y, z) + C.$$

Consequently, there exist $v_{(y,z)} \in F(\widehat{x}, y, z)$ and $c \in C$ satisfying

$$(1-t)w_y = v_{(y,z)} + c.$$

Proceeding to the next step, we have

$$v_{(y,z)} = -c + (1-t)w_y \in -C - \text{int}\, C \subset -\text{int}\, C,$$

and then

$$F(\widehat{x}, y, z) \not\subseteq Z \setminus (-\text{int}\, C), \quad \forall z \in (\widehat{x}, \overline{u}], \quad \forall y \in T(\widehat{x}). \tag{3.4}$$

Taking $\overline{z} \in (\widehat{x}, \overline{u}] \cap E : \overline{z} = \overline{t}\widehat{x} + (1-\overline{t})\overline{u}$ and $\overline{t} \in (0, 1)$, by (3.4), we get

$$F(\widehat{x}, y, \overline{z}) \not\subseteq Z \setminus (-\text{int}\, C), \quad \forall y \in T(\widehat{x}),$$

which contradicts (3.3).

Let $\widehat{x} \notin \text{core}_K E$. By assumption (vi) of the theorem, there exists $\widehat{u} \in \text{core}_K E$ such that

$$F(\widehat{x}, y, \widehat{u}) \not\subseteq Z \setminus (-C), \quad \forall y \in T(\widehat{x}),$$

then for each $y \in T(\widehat{x})$, there exists $m_y \in F(\widehat{x}, y, \widehat{u})$ such that $m_y \in -C$. Take arbitrarily $y \in T(\widehat{x})$ and $z \in (\widehat{u}, \overline{u}]$: $z = t\widehat{u} + (1-t)\overline{u}$ and $t \in [0, 1)$. From the C-convexity of $F(\widehat{x}, y, \cdot)$, we have

$$tm_y + (1-t)w_y \in tF(\widehat{x}, y, \widehat{u}) + (1-t)F(\widehat{x}, y, \overline{u}) \subset F(\widehat{x}, y, z) + C.$$

Following the same argument as above, we get that (3.4) holds for all $z \in (\widehat{u}, \overline{u}]$ and for all $y \in T(\widehat{x})$. Getting $\overline{z} \in (\widehat{u}, \overline{u}] \cap E : \overline{z} = \overline{t}\widehat{u} + (1-\overline{t})\overline{u}$ and $\overline{t} \in (0, 1)$, by (3.4), we have

$$F(\widehat{x}, y, \overline{z}) \nsubseteq Z \setminus (-\mathrm{int}\, C), \quad \forall y \in T(\widehat{x}),$$

which contradicts (3.3).

Therefore, the assertion of Theorem 3.1 is true. □

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